Hyperbolic systems of equations and double null foliations of spacetime

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Outline

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- Brief general relativity overview

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- Double null foliations

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- Bianchi equations and general hyperbolic systems

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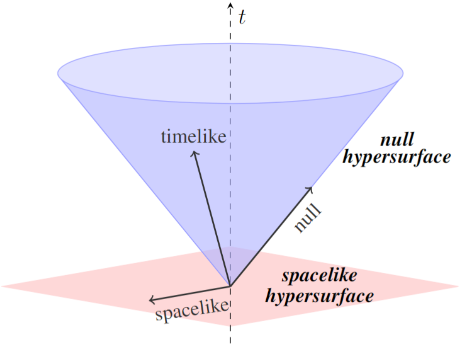
- Bianchi equations and general hyperbolic systems
- Algebraic constraints for linearized Bianchi equations

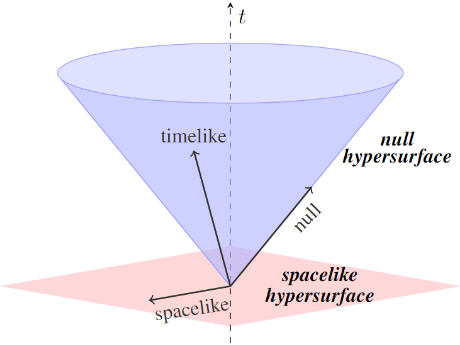
General relativity



Definition

A **spacetime** is a 4-dimensional Lorentzian manifold (M, g).

General relativity



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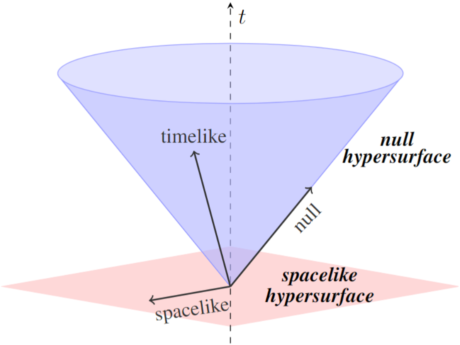

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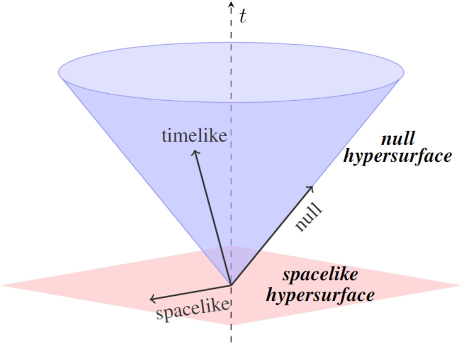
General relativity



Remark

Massless radiation (electromagnetic, gravitational,...) travels along null hypersurfaces

General relativity

Einstein equations

A Lorentzian manifold (M, g) satisfies the **Einstein equations** if

$$\mathrm{Ric}(g)_{\mu\nu} - \frac{1}{2}\mathrm{Scal}(g)g_{\mu\nu} = T_{\mu\nu}, \tag{E}$$

where $T_{\mu\nu}$ represents the matter distribution in spacetime. In particular, (M, g) satisfies the **vacuum Einstein equations** if

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Bianchi equations

In a vacuum spacetime, the Bianchi equations for the curvature are

$$\nabla_{[\alpha} W_{\beta\gamma]\delta\rho} := \nabla_\alpha W_{\beta\gamma\delta\rho} + \nabla_\beta W_{\gamma\alpha\delta\rho} + \nabla_\gamma W_{\alpha\beta\delta\rho} = 0. \tag{B}$$

Double null foliation – foliation adapted to radiation

A **double null foliation** is a choice of two “optical” functions

$$u, \underline{u} : M \to \mathbb{R}$$

whose level sets foliate the spacetime in a certain way.

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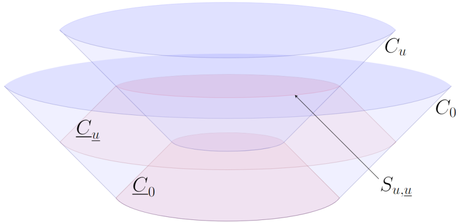
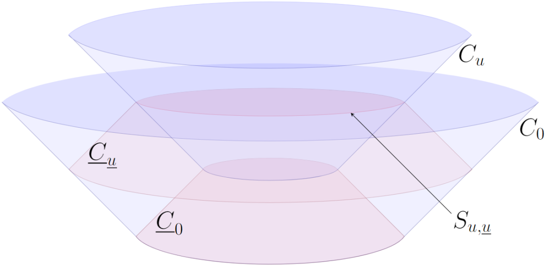


Figure: A double null foliation of a spacetime (M, g)

Double null foliation



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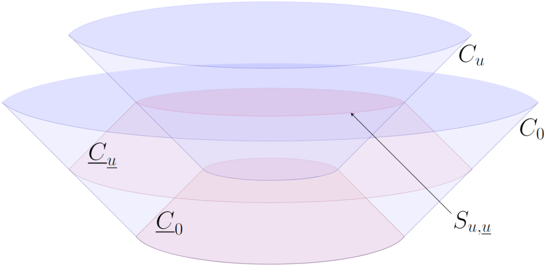

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- We let $\gamma = \gamma_{u,\underline{u}} = g|_{S_{u,\underline{u}}}$ be the (Riemannian) restriction of the spacetime metric to the spheres $S_{u,\underline{u}}$
- Write ∇ for the Levi-Civita connection of (M, g) and $\nabla\!\!\!\!/$ for the Levi-Civita connection of (S, γ)

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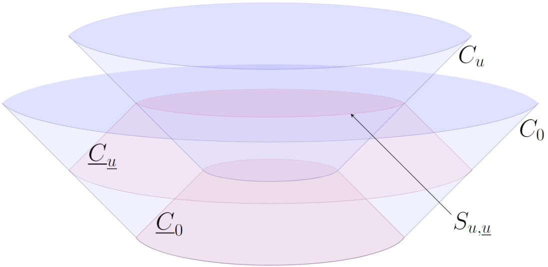

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- Incredibly powerful tool in mathematical GR (stability, singularity formation, more) [Christodoulou, Klainerman-Nicolò, Klainerman-Rodnianski-Luk, Shlapentokh-Rothman, Taylor, Dafermos-Holzegel-Rodnianski-Taylor, Oh-Luk-Shlapentokh-Rothman, many others]

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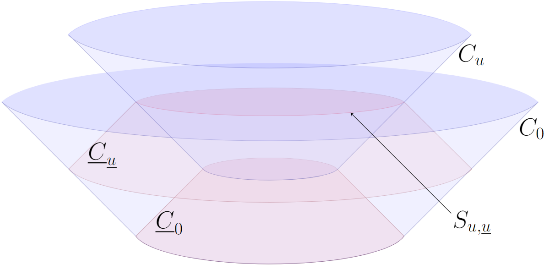

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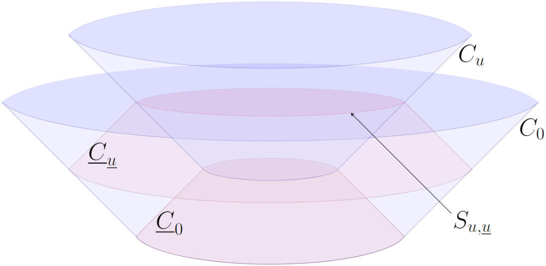

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- Well-suited for analysis of radiation (gravitational, electromagnetic,...) which *travels along null hypersurfaces*
- *Characteristic IVP:* prescribe initial data along $C_0 \cup \underline{C}_0$

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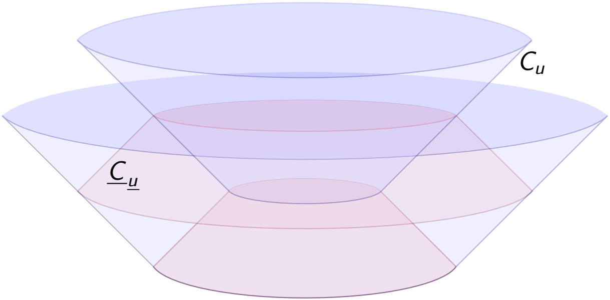
Let $e_3 = -\nabla \underline{u}, e_4 = -\nabla u$. These are null vector fields with e_3 tangent to $\underline{C}_{\underline{u}}$ and e_4 tangent to C_u.



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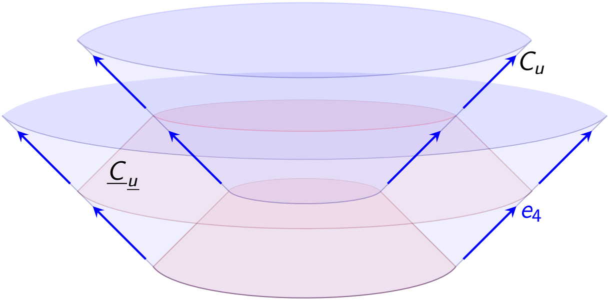

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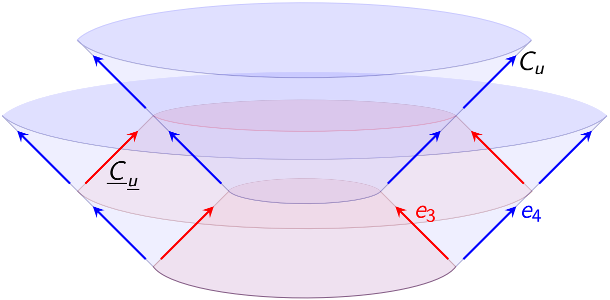

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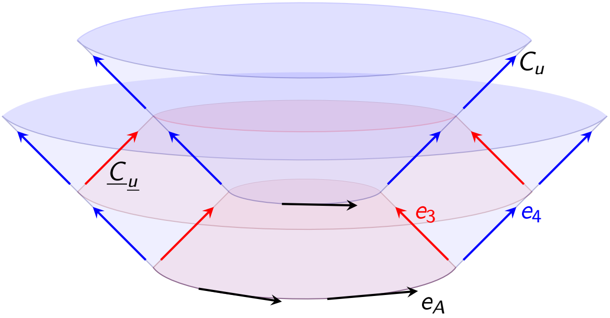

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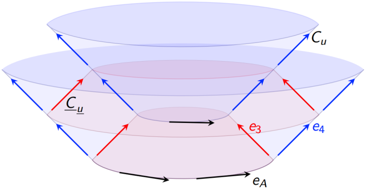
Weyl tensor components



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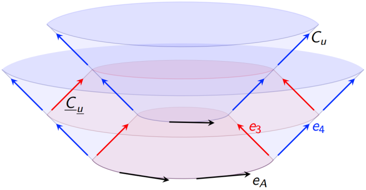

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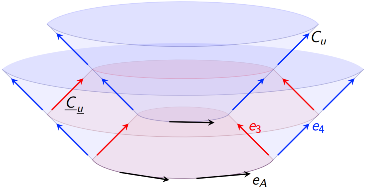

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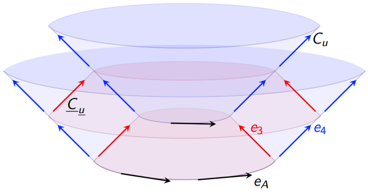

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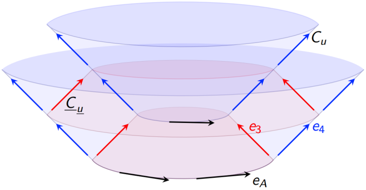

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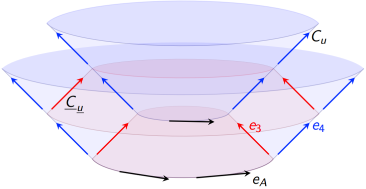

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For any S-tangent tensorfield ξ, we write $\nabla\!\!\!\!/_3\xi, \nabla\!\!\!\!/_4\xi$ denote the projections of $\nabla_3\xi, \nabla_4\xi$ (respectively) to $S_{u,\underline{u}}$.

Null Bianchi equations

Decomposing the Bianchi equations for the Weyl tensor,

$$\nabla_{[\alpha} W_{\beta\gamma]\delta\rho} = 0,$$

in the null frame $(e_\mu)_{\mu=1}^4$, gives the **null Bianchi equations.**

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$$\begin{aligned} \nabla\!\!\!/_3 \Psi[W] &= \nabla\!\!\!/ \Psi[W] + \Gamma \cdot \Psi[W] \\ \nabla\!\!\!/_4 \Psi[W] &= \nabla\!\!\!/ \Psi[W] + \Gamma \cdot \Psi[W] \end{aligned} \tag{B}$$

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These can be regarded as *propagation equations* for the Weyl tensor components $\Psi[W]$ and are used to obtain $L^p(S_{u,\underline{u}})$ and $L^p(C_u), L^p(\underline{C}_{\underline{u}})$ estimates on Ψ.

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- Expressing (VE) in a double null foliation gives the null Bianchi equations (for the curvature) & null structure equations (for the Ricci coefficients)
- These equations have excellent structure (propagation equations, hyperbolicity, elliptic equations on $S_{u,\underline{u}}$)
- Even in works which don't explicitly use a double null foliation, null Bianchi & null structure used widely [Christodoulou-Klainerman, Zipser, Bieri,...]

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\end{aligned}$$

Null Bianchi equations

Up to lower-order terms, the null Bianchi equations are:

$$\begin{aligned} \not\nabla_3 \alpha[W] &\approx \not\nabla \hat{\otimes} \beta[W] & \not\nabla_4 \beta[W] &\approx \not{\mathrm{d}}\mathrm{iv}\, \alpha[W] \\ \not\nabla_3 \beta[W] &\approx \not\nabla \rho[W] + {}^*\not\nabla \sigma[W] & \not\nabla_4 (\rho[W], \sigma[W]) &\approx (\not{\mathrm{d}}\mathrm{iv}\, \beta[W], -\mathrm{c}\not{\mathrm{u}}\mathrm{rl}\, \beta[W]) \\ \not\nabla_3 (\rho[W], \sigma[W]) &\approx (-\not{\mathrm{d}}\mathrm{iv}\, \underline{\beta}[W], -\mathrm{c}\not{\mathrm{u}}\mathrm{rl}\, \underline{\beta}[W]) & \not\nabla_4 \underline{\beta}[W] &\approx -\not\nabla \rho[W] + {}^*\not\nabla \sigma[W] \\ \not\nabla_3 \underline{\beta}[W] &\approx -\not{\mathrm{d}}\mathrm{iv}\, \underline{\alpha}[W] & \not\nabla_4 \underline{\alpha}[W] &\approx -\not\nabla \hat{\otimes} \underline{\beta}[W] \end{aligned}$$

Notation: $\not\nabla$ is the induced connection on the spheres $S_{u,\underline{u}}$, $\not{\mathrm{d}}\mathrm{iv}$ is the divergence operator of $S_{u,\underline{u}}$, $\not\nabla \hat{\otimes}$ is the traceless part of the symmetrized covariant derivative, $*$ denotes the Hodge dual on the $S_{u,\underline{u}}$. See [Christodoulou (2008)].

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Key point: Bianchi pairs. The principal part of equations on the *right* is (minus) the $L^2(S_{u,\underline{u}})$-adjoint of the principal part of the equations on the *left*, for example

$$\int_{S_{u,\underline{u}}} \phi \cdot \mathrm{d}\!\!/\mathrm{iv}\,\alpha \, d\mu_{S_{u,\underline{u}}} = -\int_{S_{u,\underline{u}}} \nabla\!\!\!/\hat{\otimes}\phi \cdot \alpha \, d\mu_{S_{u,\underline{u}}} \quad \forall \phi \in C^\infty(S_{u,\underline{u}}).$$

Double null hyperbolic systems

- The $L^2(S_{u,\underline{u}})$-adjoint property gives the null Bianchi equations their hyperbolicity [Taylor (2017)]

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Double null hyperbolic system (DNH)

$$\nabla\!\!\!/_3 \Psi^{(i)} = \mathcal{D}\!\!\!/_{\Psi^{(i)}} \underline{\Psi}^{(1)} + E^{(i)}, \qquad \nabla\!\!\!/_4 \underline{\Psi}^{(i)} = \mathcal{D}\!\!\!/_{\underline{\Psi}^{(i)}} \Psi^{(i)} + \underline{E}^{(i)} \qquad (i = 1, \ldots, N)$$

Notation: $E^{(i)}, \underline{E}^{(i)}$ are "lower-order" terms; $\mathcal{D}\!\!\!/_{\Psi^{(i)}}, \mathcal{D}\!\!\!/_{\underline{\Psi}^{(i)}}$ are first-order differential operators on $S_{u,\underline{u}}$ (e.g. $\mathrm{d}\!\!\!/\mathrm{iv}, \nabla\!\!\!/$, etc.) which are "Bianchi-paired", i.e. they are $L^2(S_{u,\underline{u}})$-anti-adjoints of each other:

$$\int_{S_{u,\underline{u}}} \phi \cdot \mathcal{D}\!\!\!/_{\Psi^{(i)}} \psi \, d\mu_{S_{u,\underline{u}}} = - \int_{S_{u,\underline{u}}} \mathcal{D}\!\!\!/_{\underline{\Psi}^{(i)}} \phi \cdot \psi \, d\mu_{S_{u,\underline{u}}} \quad \forall \phi, \psi \in C^\infty(S_{u,\underline{u}}).$$

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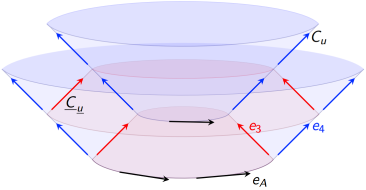

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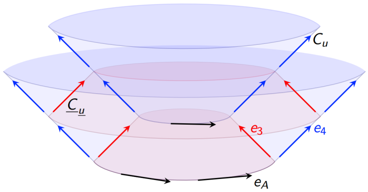

Characteristic IVP: Prescribe initial data $\Psi_0^{(i)}$ along C_0 and $\underline{\Psi}_0^{(i)}$ along $\underline{C}_0$

Aside: Bianchi pairing and energy estimates

The Bianchi pairing is essential to obtaining energy estimates for the system. Define the energy $\mathcal{E}$:

$$\mathcal{E}[\Psi](u, \underline{u}) := \frac{1}{2} \int_{S_{u,\underline{u}}} |\Psi|^2 \, d\mu_{S_{u,\underline{u}}}.$$

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&= -\int_{S_{u,\underline{u}}} \nabla\!\!\!/_4 \underline{\Psi} \cdot \Psi \, d\mu_{S_{u,\underline{u}}} + \text{lower order}
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&= -\int_{S_{u,\underline{u}}} \nabla\!\!\!/_4 \underline{\Psi} \cdot \Psi \, d\mu_{S_{u,\underline{u}}} + \text{lower order} \\
&= -\nabla\!\!\!/_4 \mathcal{E}[\underline{\Psi}](u, \underline{u}) + \text{lower order.}
\end{aligned}$$

Aside: Bianchi pairing and energy estimates

Integrating

$$\nabla\!\!\!/_3 \mathcal{E}[\Psi](u, \underline{u}) + \nabla\!\!\!/_4 \mathcal{E}[\Psi](u, \underline{u}) = \text{lower order}$$

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gives

Energy estimates for DNH

$$\int_{C_u} |\Psi|^2 \, d\mu_{C_u} + \int_{\underline{C}_{\underline{u}}} |\underline{\Psi}|^2 \, d\mu_{\underline{C}_{\underline{u}}} \lesssim \text{Initial Data.}$$

Double null hyperbolic systems

$$
\begin{aligned}
\nabla\!\!\!/_3 \alpha[W] &\approx \nabla\!\!\!/ \hat{\otimes} \beta[W] & \nabla\!\!\!/_4 \beta[W] &\approx \mathrm{d}\!\!/\mathrm{iv}\, \alpha[W] \\
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- The Bianchi equations *almost* form a DNH.

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- Other than $\alpha[W], \underline{\alpha}[W]$, each Weyl tensor component satisfies two independent equations
- Fix: choose a subset of the equation to be *constraints* and the others to be *evolution* equations
- Then hope that under the flow of the remaining equations, these constraints propagate if satisfied by the initial data (more on this later)

Linearized Bianchi equations

Fix a background vacuum spacetime (M, g). In (B) replace Weyl tensor components $\alpha[W], \beta[W]$, etc. with unknowns α, β, etc. which are to be solved for.

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$$\nabla\!\!\!/_3 \beta = \nabla\!\!\!/ \rho + {}^*\nabla\!\!\!/ \sigma + \Gamma \cdot \Psi \qquad \nabla\!\!\!/_4(\rho, \sigma) = (\mathrm{d}\!\!/\mathrm{iv}\,\beta, -\mathrm{c}\!\!/\mathrm{url}\,\beta) + \Gamma \cdot \Psi$$

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Linearized Bianchi equations: evolution and constraints

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Linearized Bianchi equations: evolution and constraints

- **Question:** How do we choose which are *constraint equations* and which are *evolution equations*?
- Choose to be "as hyperbolic as possible"
- Starting point: $\alpha, \underline{\alpha}$ have only one equation each, so to be well-posed these *must* be viewed as evolution equations
- Unfortunately, there is *no way* to choose evolution equations in such a way that (1) every unknown has a Bianchi-paired evolution equation and (2) the system is not overdetermined

Linearized Bianchi equations: evolution and constraints

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$$\nabla\!\!\!/_3\underline{\beta} = -\mathrm{d}\!\!\!/\mathrm{iv}\underline{\alpha} + \Gamma\cdot\Psi$$

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Evolution equations

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Constraint equations

Linearized Bianchi equations: evolution and constraints

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Linearized Bianchi equations: evolution and constraints

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Linearized Bianchi equations: evolution and constraints

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$$\begin{cases} \Xi := \nabla\!\!\!/_3\beta - \nabla\!\!\!/\rho - {}^*\nabla\!\!\!/\sigma - \Gamma\cdot\Psi & = 0 \\ (P,Q) := \nabla\!\!\!/_4(\rho,\sigma) - (\mathrm{d}\!\!/\mathrm{iv}\,\beta, -\mathrm{cu}\!\!/\mathrm{rl}\,\beta) - \Gamma\cdot\Psi & = 0 \\ \underline{\Xi} := \nabla\!\!\!/_3\underline{\beta} + \mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha} - \Gamma\cdot\Psi & = 0 \end{cases}$$

Linearized Bianchi equations: evolution and constraints

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Remark: A collection of tensor fields $\Psi := (\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha})$ satisfies (LB) if and only if it satisfies both the evolution and constraint equations.

Propagation of constraints

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$$\mathcal{V}_p = \left\{ \begin{matrix} \text{symmetric traceless} \\ \text{2-covariant tensors} \end{matrix} \right\} \times T_p^* S_{u,\underline{u}} \times \mathbb{R} \times \mathbb{R} \times T_p^* S_{u,\underline{u}} \times \left\{ \begin{matrix} \text{symmetric traceless} \\ \text{2-covariant tensors} \end{matrix} \right\}$$

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Theorem (S.)

Fix a background spacetime (M, g) *admitting a double null foliation. There is a pointwise linear map* $\mathfrak{L}_W|_p : \mathcal{V}_p \to T^*_p S_{u,\underline{u}} \times T^*_p S_{u,\underline{u}} \times \mathbb{R} \times \mathbb{R}$*, depending only on the Weyl tensor of* (M, g) *such that any solution of* (LB) satisfies

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Moreover, if (M, g) is not Minkowski spacetime, then

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That is, solutions to (LB) must lie in a codimension ≥ 1 subspace of $\mathcal{V}$.

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If $\alpha[W]$ *or* $\underline{\alpha}[W]$ is nonzero, then we can write a basis for the $\alpha, \underline{\alpha}$ part of $\ker \mathfrak{L}_W$:

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Very constrained!

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- Obtain many terms

Algebraic constraints: proof idea

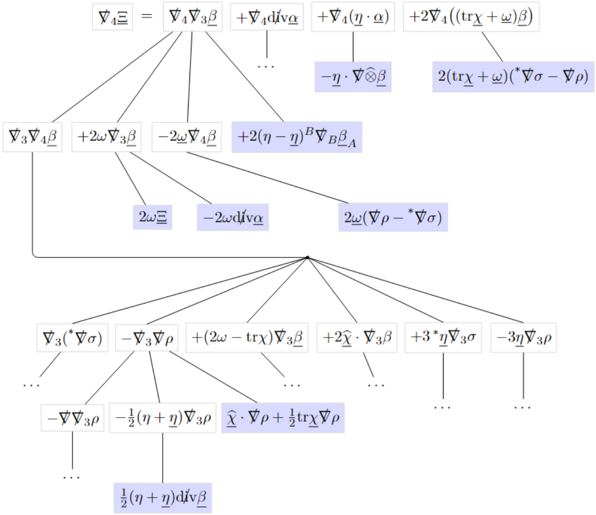

Figure: Top-order expansion of $\slashed{\nabla}_4\underline{\Xi}$

Algebraic constraints: proof idea

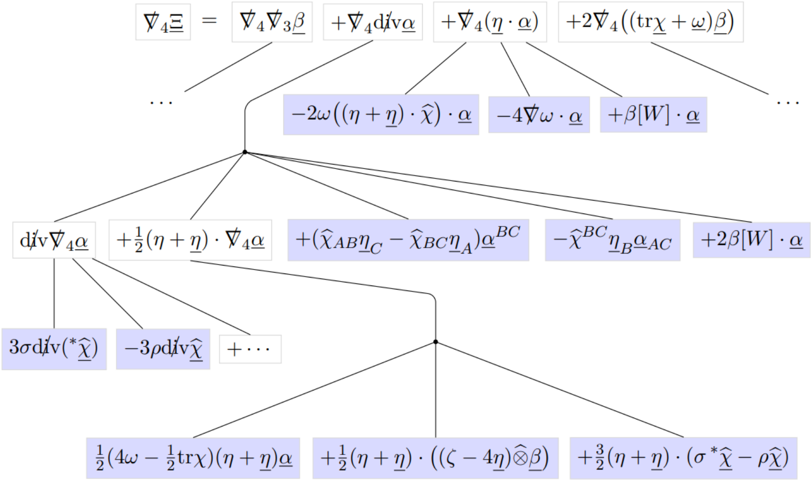

Figure: Lowest-order expansion of $\nabla\!\!\!/_4\underline{\Xi}$

Algebraic constraints: proof idea

- After grouping terms, obtain for every differential constraint $\Xi, \underline{\Xi}, P, Q$ a differential equation of the form

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Algebraic constraints – consequences and further questions

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Thank You!